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and Studies of Monetary Policy**

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# Notes on Agents' Behavioral Rules Under Adaptive Learning and Studies of Monetary Policy<sup>\*</sup>

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## ABSTRACT

These notes try to clarify some discussions on the formulation of individual intertemporal behavior under adaptive learning in representative agent models. First, we discuss two suggested approaches and related issues in the context of a simple consumption-saving model. Second, we show that the analysis of learning in the NewKeynesian monetary policy model based on “Euler equations” provides a consistent and valid approach.

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# 1 Introduction

In the literature on adaptive learning in infinite horizon representative agent settings it is often assumed that agents base their behavior on an Euler equation that is derived under subjective expectations.<sup>1</sup> This formulation has sometimes been criticized in that it does not require that the intertemporal budget constraint be satisfied for the agent since the constraint is not explicitly used when deriving the behavioral rule of the agent.<sup>2</sup>

Another point of criticism has been that the formulation is not natural since it postulates that agents are making forecasts of their future consumption, which is their own choice variable. Preston (2005) has proposed an interesting reformulation of (linearized) intertemporal behavior under learning in which agents are assumed to incorporate a “subjective version” of their intertemporal budget constraint in their behavior under learning. A further issue sometimes raised, is whether temporary equilibrium equations based on Euler equations with subjective expectations, such as those used in Bullard and Mitra (2002) and Evans and Honkapohja (2003), are subject to inconsistency when subjective expectations are used in equilibrium equations that have normally been derived under rational expectations.

In these notes we first clarify the relationship between two formulations of intertemporal behavior under adaptive learning and show that the intertemporal accounting consistency holds in an *ex post* sense along the sequence of temporary equilibria under “Euler equation” learning. This is done in the simple context of a consumption-saving model. Second, we consider the Preston (2005) model of monetary policy under learning and show that, under plausible assumptions, the usual system based on Euler equations with subjective expectations can be obtained from Preston’s approach and is, therefore, a valid way of analyzing learning dynamics under incomplete knowledge.

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<sup>1</sup>This is done e.g. in some of the models of Chapter 10 of Evans and Honkapohja (2001). See also the discussion in Marcet and Sargent (1989) and Sargent (1993).

<sup>2</sup>This point has been made in the context of New Keynesian models of monetary policy. The approach based on Euler equations is used e.g. in Bullard and Mitra (2002) and Evans and Honkapohja (2003).

## 2 A Permanent Income Model

Consider a model in which income follows an exogenous process and there is a representative consumer who makes consumption-saving decisions.<sup>3</sup> The consumer has a standard intertemporal utility function

$$\hat{E}_t \sum_{s=t}^{\infty} \beta^{s-t} U(C_s) \quad (1)$$

and the accounting identity for net assets  $W_s$  is

$$W_{s+1} = R_s W_s - C_s + Y_s. \quad (2)$$

For the initial period of the economy net assets are taken to be zero, i.e.  $W_t = 0$ .<sup>4</sup>  $R_s$  is the one-period real gross rate of return factor for a safe one-period loan, assumed known at  $s$ . Because we are in a general equilibrium framework we do not take it to be fixed and its value will be determined by market clearing. Output  $Y_s$  follows an exogenous process

$$Y_s = M Y_{s-1}^\rho V_s \quad (3)$$

or

$$\log Y_s = \mu + \rho \log Y_{s-1} + v_s,$$

where  $|\rho| < 1$  and  $v_s$  is white noise. Expectations are not necessarily rational, which is indicated by  $\hat{\cdot}$  in the expectations operator. There is also an intertemporal budget constraint of the form

$$C_t + \sum_{s=t+1}^{\infty} \mathcal{R}_{t+1,s} C_s = Y_t + \sum_{s=t+1}^{\infty} \mathcal{R}_{t+1,s} Y_s, \quad (4)$$

where  $\mathcal{R}_{t+1,s} = (R_{t+1} \dots R_s)^{-1}$  is the market discount factor.

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<sup>3</sup>The results remain unchanged if it is assumed instead that there is finite (or infinite) number of consumers with identical characteristics, including their forecasts and learning rules.

<sup>4</sup>Note that this is a very simple general equilibrium model of a closed economy. Thus there cannot be any net paper assets (like bonds) before the economy starts.

Maximizing (1) subject to (4) yields the Euler equation as a necessary condition. It has the familiar form

$$U'(C_t) = \beta R_t \hat{E}_t U'(C_{t+1}) \quad (5)$$

and in equilibrium  $C_t = Y_t$ , as output is assumed to be perishable. In this temporary equilibrium framework, agents' demand for consumption goods  $C_t$  depends on their forecast  $\hat{E}_t U'(C_{t+1})$  and on the interest rate factor  $R_t$ , in accordance with (5). Imposing the market clearing condition  $C_t = Y_t$  we see that (5) determines the interest rate according to

$$R_t^{-1} = \beta(\hat{E}_t U'(C_{t+1}))/U'(Y_t).$$

This gives us the temporary equilibrium at  $t$ .

We now log-linearize (5) at a non-stochastic steady state. Standard computations yield

$$c_t = \hat{E}_t c_{t+1} - \sigma r_t, \quad (6)$$

where  $c_t = \log(C_t/\bar{C})$ ,  $r_t$  is the net return, based on the approximation  $r_t \approx \log(R_t/\bar{R})$  and  $\sigma = -\frac{U'(\bar{C})}{U''(\bar{C})\bar{C}}$  is the coefficient of intertemporal substitution (or risk aversion). (6) is the consumer's demand schedule giving current consumption demand as a function of the interest rate  $r_t$  and forecasts about the next period.

The log-linearization of the output process gives

$$y_t = \rho y_{t-1} + v_t, \quad (7)$$

where  $y_t = \log(Y_t/\bar{Y})$ . (Bars over the variables denote the non-stochastic steady state.) The rational expectations equilibrium (REE) of the linearized model is given by

$$r_t = -(1 - \rho)\sigma^{-1}y_t$$

and for rational forecasts we have

$$E_t c_{t+1} = \rho y_t. \quad (8)$$

## 2.1 Learning Based on Euler Equations

To formulate learning in terms of the linearized Euler equation (6), which we will call EE approach subsequently, we suppose that agents are learning, using a PLM corresponding to the REE:

$$\hat{E}_t c_{t+1} = m_t + n_t y_t, \quad (9)$$

where  $(m_t, n_t)$  are obtained using a regression of  $c_s$  on  $y_{s-1}$  using data  $s = 1, \dots, t-1$ . The data are then used to update parameter estimates to  $(m_{t+1}, n_{t+1})$  and we proceed to period  $t + 1$ .

Note that the rational forecast function (8) is a particular case of (9) and the basic question is whether  $(m_t, n_t) \rightarrow (0, \rho)$  over time. This can easily be verified, for example using E-stability arguments.<sup>5</sup> Suppose we have (9) where the time subscripts are dropped from the parameters, i.e.  $\hat{E}_t c_{t+1} = m + n y_t$ . Temporary equilibrium, given forecasts  $\hat{E}_t c_{t+1}$ , in the linearized model is

$$r_t = -\sigma^{-1}(y_t - \hat{E}_t c_{t+1}) = -\sigma^{-1}[y_t(1 - n) - m]$$

and the ALM is

$$T(m, n) = (0, \rho).$$

The E-stability differential equations are thus

$$\frac{d(m, n)}{d\tau} = (0, \rho) - (m, n),$$

which yields convergence of adaptive learning in this model.

Is this a plausible formulation? One of the necessary conditions for individual optimization is on the margin between today's consumption and tomorrow's consumption, and implementation of this FOC requires a forecast of that agent's own  $C_{t+1}$ . It might seem odd to have an agent forecasting his own behavior, but it is actually very natural. In the REE future consumption is related to the key exoge-

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<sup>5</sup>For the connection between least squares learning and E-stability see Evans and Honkapohja (2001).

nous state variable (e.g. income in the model of consumption). In a temporary equilibrium with learning agents are just trying to infer this relationship from past data and in forecasting they use the estimated relationship. The agent needs to plan what level of consumption he will choose in the following period and he also considers the perceived relation of consumption to the key exogenous variable. His best guess, given the AR(1) income process, is plausibly a linear function of current income. Thinking a single step ahead, in this way, appears to us to be one plausible and natural form of bounded rationality.

Note that, at first sight, this formulation of agent's behavior rule does not seem to require explicitly the intertemporal life-time budget constraint (4) or transversality condition. Yet it is not inconsistent with such a constraint as the agent can be thought to solve the intertemporal problem under subjective expectations. When the behavior rule of the agent is based on the Euler equation, only the one-step forward margin, the flow budget constraint and one-step forecasts are explicitly used.<sup>6</sup>

A boundedly rational agent making use only of the current Euler equation and an appropriate forecast function will converge to the household optimum under least squares learning. It can, moreover, be shown that, along the sequence of temporary equilibria during the convergent learning, *ex post* consistency in the accounting over the infinite horizon is fulfilled. To see this we note that, iterating the flow accounting identity, we have

$$C_t + \sum_{s=t+1}^T \mathcal{R}_{t+1,s} C_s = Y_t + \sum_{s=t+1}^T \mathcal{R}_{t+1,s} Y_s + \mathcal{R}_{t+1,T} W_{T+1}.$$

In the sequence of temporary equilibria  $C_s = Y_s$  for all  $s$ , which implies that  $\mathcal{R}_{t+1,T} W_{T+1} = 0$  and so the *ex post* transversality condition must hold. If learning is convergent, then intertemporal consistency is achieved. Once the EE learning has reached the REE, the agent has the correct forecast function (8) and his behavior based on the Euler equation generates the REE sequence  $(c_s^*, r_s^*)$  of consumptions

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<sup>6</sup>Note also that, in many derivations of the REE, the intertemporal budget constraint is checked only at REE prices. Indeed, there could be problems with existence of solutions to household optimum at arbitrary prices sequences.

and interest rates. This type of behavior by the agent is then consistent with full intertemporal optimization since if he is faced with the sequence of interest rates  $r_s^*$  he would choose the consumption sequence  $c_s^*$  which does satisfy the transversality condition.<sup>7</sup>

In other economic models, learning based on Euler equations may fail to be stable. In cases of instability one could argue that if the economy diverges along an explosive path, the household would begin to think through the implications of its lifetime budget constraint and/or transversality condition and eventually alter its behavior. Of course, in the divergent case the log-linearization is also invalid since the economy will not stay near the steady state.

## 2.2 Learning With Perceptions Over an Infinite Horizon

A different form of learning behavior is developed by Preston (2005) in the context of a New Keynesian model of monetary policy.<sup>8</sup> His approach can also be simply presented in the current context. The starting point is to log-linearize the intertemporal budget constraint (4) at the non-stochastic steady state, which yields

$$\hat{E}_t \bar{C} c_t + \sum_{s=t+1}^{\infty} \bar{\mathcal{R}}_{t+1,s} \bar{C} \hat{E}_t c_s = \bar{Y} y_t + \sum_{s=t+1}^{\infty} \bar{\mathcal{R}}_{t+1,s} \bar{Y} \hat{E}_t y_s, \quad (10)$$

where in fact  $\bar{\mathcal{R}}_{t+1,s} = (1/\bar{R})^{s-t} = \beta^{s-t}$  and  $\bar{C} = \bar{Y}$  at the steady state. Next, we iterate the linearized Euler equation (6) backwards for  $s \geq t+1$ , giving

$$\hat{E}_t c_s = c_t + \sigma \sum_{j=t}^{s-1} \hat{E}_t r_j. \quad (11)$$

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<sup>7</sup>EE learning is a special case of shadow-price learning, which can be shown to deliver asymptotically optimal decision-making in general settings. See Evans and McGough (2010).

<sup>8</sup>Infinite-horizon learning based on an iterated Euler equation was applied to the “investment under uncertainty” example in pp. 122-125 of Sargent (1993).



Substituting (11) into (10) leads to

$$c_t + \sum_{s=t+1}^{\infty} \beta^{s-t} [c_t + \sigma \sum_{j=t}^{s-1} \hat{E}_t r_j] = y_t + \sum_{s=t+1}^{\infty} \beta^{s-t} \hat{E}_t y_s.$$

Rearranging the summation and manipulation give a linearized consumption function in the form

$$c_t = \sum_{s=t}^{\infty} \beta^{s-t} [(1 - \beta) \hat{E}_t y_s - \sigma \beta \hat{E}_t r_s]. \quad (12)$$

We will call this the infinite horizon (IH) approach to modeling adaptive learning by the agent.

There are several important comments about this formulation.

First, note that if (12) is the behavioral rule of the learning agent, then the agent must make forecasts about future income/output and rates of return into the infinite future. The agent is thus assumed to be very far-sighted even though he is boundedly rational.

Second, it can be asked whether the EE approach is consistent with (12). This is naturally the case, since the derivation of (12) relies in part on (6). Moreover, advancing (12) and multiplying by  $\beta$  one period gives

$$\beta c_{t+1} = \sum_{s=t+1}^{\infty} \beta^{s-t} [(1 - \beta) \hat{E}_{t+1} y_s - \sigma \beta \hat{E}_{t+1} r_s],$$

to which one can apply the subjective expectations  $\hat{E}_t(\cdot)$ . Once this has been done, it is seen that

$$c_t = (1 - \beta) y_t - \sigma \beta r_t + \beta \hat{E}_t c_{t+1},$$

so that by using market clearing  $c_t = y_t$  the Euler equation (6) also obtains.

This derivation presumes that the law of iterated expectations holds for the subjective expectations of the agent. For standard formulations of adaptive learning this is usually assumed. For example, suppose that agents do not know the relationship between  $y_t$  and  $r_t$  and assume that the return  $r_t$  is a linear function of the key state

variable  $y_t$ , so that at time  $t$  they have the PLM

$$r_t = d_t + f_t y_t. \quad (13)$$

For simplicity, we assume that they know the true process of  $y_t$ , (7). The agents' forecasts are assumed to behave as follows

$$\hat{E}_t \hat{E}_{t+1} r_s = \hat{E}_t (d_{t+1} + f_{t+1} \hat{E}_{t+1} y_s) = d_t + f_t \hat{E}_t y_s,$$

which says that in iterating expectations back to an earlier period the point estimates of the PLM parameters are shifted back to the earlier values.<sup>9</sup> This is the standard formulation in the adaptive learning literature, and can be viewed as an axiom of the approach.

Third, it is of interest to consider whether learning using the forecasts based on (13) converges. We again study this using E-stability, so that the PLM is  $r_t = d + f y_t$ . Then (12) can be written as

$$\begin{aligned} c_t &= \sum_{s=t}^{\infty} \beta^{s-t} [(1 - \beta) \hat{E}_t y_s - \sigma \beta \hat{E}_t r_s] \\ &= \sum_{s=t}^{\infty} \beta^{s-t} \{[(1 - \beta) - \sigma \beta f] \hat{E}_t y_s - \sigma \beta d\}. \end{aligned}$$

We have

$$\sum_{s=t}^{\infty} \beta^{s-t} \hat{E}_t y_s = \sum_{s=t}^{\infty} \beta^{s-t} \rho^{s-t} y_t = \frac{1}{1 - \beta \rho} y_t$$

and we get

$$c_t = \frac{1 - \beta - \sigma \beta f}{1 - \beta \rho} y_t - \frac{\sigma \beta d}{1 - \beta}.$$

The temporary equilibrium value of the rate of return is determined from the Euler

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<sup>9</sup>More generally, one could have the agents also learn the parameters of the process for  $y_t$ . Then they would also have a PLM of the form  $y_t = a_t + b_t y_{t-1} + v_t$ . In this case the iterated expectations would take the form  $\hat{E}_t(a_{t+1} + b_{t+1} \hat{E}_{t+1} y_{s-1}) = a_t + b_t \hat{E}_t y_{s-1}$ .

equation (6), so that

$$r_t = -\sigma^{-1}(y_t - \hat{E}_t c_{t+1}) = -\sigma^{-1}(y_t - \frac{1 - \beta - \sigma\beta f}{1 - \beta\rho} \rho y_t + \frac{\sigma\beta d}{1 - \beta}).$$

The T-mapping is thus

$$\begin{aligned} d &\rightarrow -\frac{\beta d}{1 - \beta} \\ f &\rightarrow -\sigma^{-1}(1 - \frac{1 - \beta - \sigma\beta f}{1 - \beta\rho} \rho). \end{aligned}$$

The differential equation defining E-stability consists of two independent linear equations with negative coefficients on the variables  $d$  and  $f$ , respectively and so we have E-stability.

## 2.3 Further Discussion

Comparing the two approaches to agent's behavior under learning we see that the EE approach has the agents making forecasts only one period ahead. It is thus assumed that the agent is relatively short sighted. In contrast, in the IH approach the agent has to make forecasts over the entire infinite future. Thus the agent is very far-sighted. These two approaches represent different ways of modeling agent's behavior under adaptive, boundedly rational learning.

It should be noted that, quite naturally, the agent forecasts different quantities in the EE and IH approaches. Thus the natural PLM have different parameters and the respective mappings from the PLM to the ALM are also different. We have seen that the two approaches are not inconsistent in the sense that it is possible to derive the EE formulation from the IH approach under certain plausible conditions. We have convergence of learning for both approaches in this model. In terms of the degree of farsightedness the two approaches represent extreme cases. In the EE approach the boundedly rational agents look ahead only for one period while in the IH approach they look ahead into the infinite future.

Which approach is more plausible (suitable)? This will, of course, depend on

the type of situation being analyzed in an economic model. There are certainly circumstances where it would be more plausible to assume that agents have long horizons. For instance, assume that future changes in fiscal policy are announced by the government and these changes are viewed as credible by economic agents. The EE approach may not be suitable for this analysis since agents look only one period ahead and would not react to the announcement until the moment the policy change actually takes place! Normally, one would expect agents' current (short-term) decisions to be affected by the possibility of future changes since agents are assumed to be (subjective) dynamic optimizers where the horizon in their utility maximization problem is infinite (in the same spirit as RE). Since the learning analysis based on EE only requires agents to make one period ahead forecasts, these forecasts will potentially not be affected by the announcement of future policy changes.<sup>10</sup>

The IH approach is used in Evans, Honkapohja, and Mitra (2009) to analyze announced future policy changes; they consider a simple competitive representative-agent endowment economy in which the government purchases output for current consumption and levies lump-sum taxes. The baseline case has balanced-budget spending changes (which agents are assumed to know) and analyzes the dynamics arising from credible, permanent anticipated changes in government spending/taxation. Evans, Honkapohja, and Mitra (2009) utilize the consumption function of the representative agent which relates current consumption of the household to future (subjective) forecasts of taxes and real interest rates. This allows agents to react to future policy changes (in taxes) through their current consumption/savings choice. Agents need to forecast future real interest rates and taxes to decide their current consumption plans. Since the policy change is credible and announced, agents are endowed with knowledge of the future path of taxes. This announced change in future taxes leads immediately to a change in current consumption. Knowledge of the

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<sup>10</sup>Note that, under RE, it does not matter whether one conducts the analysis of policy changes using the consumption Euler equation or the consumption function (which involves IH forecasts). However, in the presence of incomplete knowledge of agents, it typically matters whether the analysis is conducted using the consumption EE or the consumption function. For instance, the latter may determine consumption levels based on interest, wage, and tax forecasts whereas the EE only involves one-period ahead interest rate (and own consumption) forecasts.

overall structure of the economy nevertheless remains incomplete, and agents must forecast future interest rates using only the history of interest rate data, revising these forecasts over time in line with standard adaptive learning models.

The results are striking. In line with the agents’ forward-looking behaviour under learning, there are immediate changes in the interest rate, with continued evolution over time, and these dynamics are in a sharp contrast to the fully rational path; see Figure 1 of their paper.<sup>11</sup>

In judging the approaches one must perhaps also take note of the empirical observation that in reality public and private forecasting institutions have only a limited time horizon, often at most two years, for detailed business cycle forecasting. Very long term projections are also made by forecasting institutions but these projections are very broad as they usually show only long term trends of relatively few variables. Perhaps the “right” approach is in between these two extremes from a practical point of view.<sup>12</sup>

### 3 Learning and Monetary Policy

The learning viewpoint has been extensively used in the past decade to analyze monetary policy design in the basic New Keynesian model presented in Woodford (2003) (see eg. Ch. 2-4) and Woodford (1996). The seminal papers of Bullard and Mitra (2002) and Evans and Honkapohja (2003) examined the performance of various Taylor type and optimal interest rate policies of the central bank using the Euler Equation (EE) approach. Preston (2005) considers a model of monetary policy using the IH approach. He shows that if the central bank uses the contemporaneous data Taylor type rule, then the learning dynamics are E-stable if and only if the Taylor principle is satisfied; see Proposition 2 of the paper. Note that this E-stability result is the same as that in Bullard and Mitra (2002).

We now demonstrate that the EE analysis of Bullard and Mitra (2002) and Evans

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<sup>11</sup>The EE approach in Evans, Honkapohja, and Mitra (2009) for the announced policy change leads to different dynamics of interest rates from the IH approach; see the paper for details.

<sup>12</sup>For a formalization of intermediate approaches, see Branch, Evans, and McGough (2010).

and Honkapohja (2003) is consistent with the IH analysis of Preston (2005), as in the simple permanent income model of Section 2. Consequently, both the EE and IH approaches are valid ways to study stability under learning in the New Keynesian setting.

### 3.1 Framework

We start with the framework presented in Section 2 of Preston which essentially uses a dynamic stochastic equilibrium model of Woodford (2003) (Ch. 2-4). Preston (2005) derives an optimal consumption rule for a representative household and an optimal pricing rule for a representative firm.<sup>13</sup> These two equations are, respectively,

$$C_t^i = \hat{E}_t^i \left\{ \sum_{T=t}^{\infty} \beta^{T-t} [(1-\beta)Y_T - \beta\sigma(i_T - \pi_{T+1}) + \beta(g_T - g_{T+1})] \right\}, \quad (14)$$

$$p_t^i = \hat{E}_t^i \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[ \frac{(1-\alpha\beta)(\omega + \sigma^{-1})}{(1 + \omega\theta)} x_T + \alpha\beta\pi_{T+1} \right] \right\}, \quad (15)$$

where  $g_t$  is an appropriate taste shock. (In some variations of the model  $g_t$  represents a government spending shock).

Under our representative agent assumption agents have identical expectations and thus consumption and price setting (for firms able to set prices) is the same across agents, i.e. for all relevant variables  $z$  we have  $\hat{E}_t^i z = \hat{E}_t z$  and thus  $C_t^i = \int_j C_t^j dj \equiv C_t$  and  $p_t^i = \int_j p_t^j dj \equiv p_t$ . Given expectations, the temporary equilibrium values of output  $Y_t$  and the inflation rate  $\pi_t$  are determined by the market clearing condition  $Y_t = C_t$  and by the relationship between the aggregate price level and prices currently being set, given by  $\pi_t = \alpha^{-1}(1-\alpha)p_t$ . The equation for  $Y_t$  is often reexpressed in terms of the output gap  $x_t = Y_t - Y_t^n$ , where  $Y_t^n$  is the natural rate of output.

Integrating (14)-(15) over  $i$  and using these relationships gives

$$x_t = \hat{E}_t \left\{ \sum_{T=t}^{\infty} \beta^{T-t} [(1-\beta)x_{T+1} - \sigma(i_T - \pi_{T+1}) + r_T^n] \right\} \quad (16)$$

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<sup>13</sup>We refer the reader to Preston (2005) for the details of these derivations.

where  $r_T^n = g_T - g_{T+1} + Y_{t+1}^n - Y_t^n$ , and

$$\pi_t = \hat{E}_t \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} [\kappa x_T + (1-\alpha)\beta\pi_{T+1}] \right\} \quad (17)$$

where

$$\kappa = (1-\alpha)\alpha^{-1}(1-\alpha\beta)(\omega + \sigma^{-1})(1 + \omega\theta)^{-1}.$$

Preston (2005) then conducts the analysis using equations (16) and (17) as the behavioral rule for households and firms.

The analysis in Bullard and Mitra (2002) and Evans and Honkapohja (2003), on the other hand, is based on the EE approach and thus starts from the following two equations

$$x_t = \hat{E}_t x_{t+1} - \sigma \left( i_t - \hat{E}_t \pi_{t+1} \right) + r_t^n, \quad (18)$$

$$\pi_t = \kappa x_t + \beta \hat{E}_t \pi_{t+1}. \quad (19)$$

We now show how to derive (18) and (19) from (14) and (15). This implies that (18) and (19) are an equally valid framework for studying learning.

### 3.2 Derivation of Aggregate Euler Equations

The key assumption that will allow us to derive (18) and (19) from (14) and (15) is that the subjective expectations of individual agents obey the law of iterated expectations, i.e. for any variable  $z$

$$\hat{E}_t^i \hat{E}_{t+s}^i z = \hat{E}_t^i z \text{ for } s = 0, 1, 2, \dots$$

As indicated above, this is a standard assumption for agents making forecasts from linear laws of motion estimated by Least Squares.

For example, in Bullard and Mitra (2002), agent  $i$  has a perceived law of motion

(PLM) of the form<sup>14</sup>

$$\begin{aligned}x_t &= a_{x,t}^i + b_{x,t}^i r_t^n + \epsilon_{xt}, \\ \pi_t &= a_{\pi,t}^i + b_{\pi,t}^i r_t^n + \epsilon_{\pi t}\end{aligned}$$

which can be used to form future forecasts for any  $T > t$ ,

$$\hat{E}_t^i x_T = a_{x,t}^i + b_{x,t}^i \hat{E}_t^i r_T^n, \quad (20)$$

$$\hat{E}_t^i \pi_T = a_{\pi,t}^i + b_{\pi,t}^i \hat{E}_t^i r_T^n. \quad (21)$$

Note that if each agent  $i$  has identical parameter estimates (and knows the persistence parameter  $\rho$  in the process of  $r_t^n$ , a simplifying assumption without any loss of generality), then the forecasts of each agent are the same, that is,  $\hat{E}_t^i = \hat{E}_t^j$  for all  $i$  and  $j$ . This, of course, implies that  $\hat{E}_t^i = \hat{E}_t$  for all  $i$  in the analysis. We emphasize that there is no need for any single agent to make this inference when forming the forecasts needed in his decision making. In other words, every agent  $i$  forms his own forecast independently of the other agents in the economy and uses this forecast in his optimal consumption or pricing rule. It follows that the optimal consumption and pricing rules of each agent given by (14) and (15) are the same, that is,  $C_t^i = C_t$  and  $p_t^i = p_t$  for all  $i$ . (In principle the rules given by (14) and (15) could vary across households/firms if the future forecasts are different across them but homogenous forecasts force them to be the same.)

As discussed before, (20) implies for  $j \geq 1$  that

$$\hat{E}_{t+j}^i x_T = a_{x,t+j}^i + b_{x,t+j}^i \hat{E}_{t+j}^i r_T^n$$

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<sup>14</sup>Evans and Honkapohja (2003) allow for an exogenous random shock to the inflation equation (19) and consequently they examine a PLM that depends on this shock. Our central points do not depend on the specific PLM, and hold also if the PLM includes lagged endogenous variables, as in Evans and Honkapohja (2006).



and when we take expectations of the above expression at time  $t$  we obtain

$$\hat{E}_t^i \hat{E}_{t+j}^i x_T = a_{x,t}^i + b_{x,t}^i \hat{E}_t^i \hat{E}_{t+j}^i r_T^n = a_{x,t}^i + b_{x,t}^i \hat{E}_t^i r_T^n = \hat{E}_t^i x_T \quad (22)$$

In other words, it is assumed that the law of iterated expectations holds at the individual level.<sup>15</sup> With assumption (22) and identical expectations across agents, one can show that, for analytical purposes, it is possible to obtain (18) from equation (14). Although there are several ways to obtain the desired results, we give a derivation that focuses on the individual Euler equation. This will reinforce points made earlier in these notes and emphasize the details of individual decision making.

We begin by taking quasi-differences of (14). Advancing (14) by one time unit, taking expectations  $\hat{E}_t^i$  of both sides, and using the law of iterated expectations, we obtain

$$\begin{aligned} C_t^i - \beta \hat{E}_t^i C_{t+1}^i &= \hat{E}_t^i [(1 - \beta)Y_t - \beta\sigma(i_t - \pi_{t+1}) + \beta(g_t - g_{t+1})], \text{ or} \\ C_t^i &= \beta \hat{E}_t^i C_{t+1}^i + (1 - \beta)(x_t + Y_t^n) - \beta\sigma(i_t - \hat{E}_t^i \pi_{t+1}) + \beta(g_t - g_{t+1}), \end{aligned} \quad (23)$$

where for simplicity we assume that  $g_t, g_{t+1}, Y_t^n$  and  $Y_{t+1}^n$  are known at  $t$ .

To implement (23) each agent must forecast their consumption next period. Market clearing and the representative agent assumption imply that  $C_t^i = Y_t$  for all  $i, t$ , i.e. consumption of each agent is in fact equal to mean/aggregate output in each period. We assume that each agent observes this equality from historical data, and thus forecasts its consumption next period by its forecast of aggregate output.<sup>16</sup> Using also  $Y_t = x_t + Y_t^n$ , for all  $t$ , we obtain

$$\hat{E}_t^i C_{t+1}^i = \hat{E}_t^i x_{t+1} + Y_{t+1}^n.$$

Here we are following the literature in assuming that  $Y_{t+1}^n$  is observable at  $t$ , in which

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<sup>15</sup>We have kept on purpose the superscript  $i$  for individuals, though the analysis assumes identical expectations.

<sup>16</sup>Note that we do not need to make any *a priori* assumption that agents know that all agents are identical, and we do not need to assume that agents make deductions based upon this.

case it is natural to assume that  $\hat{E}_t^i C_{t+1}^i$  would incorporate this information and use least squares to forecast the unknown component  $x_{t+1}$ .<sup>17</sup> Hence

$$C_t^i = \beta \hat{E}_t^i x_{t+1} + (1 - \beta)x_t + Y_t^n - \beta\sigma(i_t - \hat{E}_t^i \pi_{t+1}) + \beta r_t^n, \quad (24)$$

where  $r_t^n = g_t - g_{t+1} + Y_{t+1}^n - Y_t^n$ .

Equation (24) is our behavioral equation giving consumption demand as a function of interest rates, current income and one-step ahead forecasts of income and inflation. As discussed earlier, although (24) does not explicitly impose the lifetime budget constraint, it is a consistent and plausible way of implementing bounded rationality, which in stable systems will indeed lead to satisfaction of the intertemporal budget constraint. Finally, from market-clearing  $C_t^i = Y_t = x_t + Y_t^n$  and using  $\hat{E}_t^i x_{t+1} = \hat{E}_t x_{t+1}$  and  $\hat{E}_t^i \pi_{t+1} = \hat{E}_t \pi_{t+1}$  we arrive at the aggregate Euler equation (18).

The derivation of (19) from (15) is analogous. Taking quasi-differences of (15) and using the law of iterated expectations at the individual level leads to the individual agent Euler equation

$$p_t^i = \alpha\beta \hat{E}_t^i p_{t+1}^i + (1 - \alpha\beta)(\omega + \sigma^{-1})(1 + \omega\theta)^{-1}x_t + \alpha\beta \hat{E}_t^i \pi_{t+1}.$$

Note that in this Euler equation agent  $i$ 's expectations of future values of  $x_T$  and  $\pi_{T+1}$  are appropriated condensed into  $\hat{E}_t^i p_{t+1}^i$ , the price the firm expects to set next period if it is again a price setter. Finally, we make use of

$$p_t^i = p_t \text{ and } \pi_t = \alpha^{-1}(1 - \alpha)p_t \text{ all } t,$$

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<sup>17</sup>However, nothing hinges on this point. In more general representative agent set-ups, each agent would forecast its consumption at  $t + 1$  by a least squares regression on all relevant information variables.

which implies that<sup>18</sup>

$$\hat{E}_t^i p_{t+1}^i = \alpha(1 - \alpha)^{-1} \hat{E}_t^i \pi_{t+1}.$$

It follows that

$$p_t^i = \alpha\beta(1 - \alpha)^{-1} \hat{E}_t^i \pi_{t+1} + (1 - \alpha\beta)(\omega + \sigma^{-1})(1 + \omega\theta)^{-1} x_t. \quad (25)$$

Equation (25) is our behavioral equation giving individual price setting as a function of the current output gap and the one-step ahead forecasts of inflation. Integrating over households and using  $\pi_t = \alpha^{-1}(1 - \alpha)p_t$  we arrive at the aggregate Euler equation (19).<sup>19</sup>

Honkapohja and Mitra (2005) have considered cases in which the central bank uses its own forecasts of inflation and output (rather than private sector forecasts) in its interest rate rule. This poses no additional complication for the above derivation of the system (18) and (19) from (14) and (15), given the assumption (which we have maintained throughout) that the consumption schedule is conditioned on current interest rates, so that  $x_t, \pi_t$  and  $i_t$  are simultaneously determined in the usual way by market clearing.

### 3.3 Some Final Remarks

The EE and IH approaches to modeling agent's behavior rule are not identical and lead to different paths of learning dynamics. Thus there is in general no guarantee that the convergence conditions for the two dynamics are identical, though this happens to be the case in the permanent income model of Section 2 and is also the outcome for some interest rate rules in the New Keynesian model of monetary policy considered in Preston (2005).

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<sup>18</sup>Because there is an exact linear relation between these variables, if agents form expectations using least squares learning, the expectations  $\hat{E}_t^i p_{t+1}^i$  and  $\hat{E}_t^i \pi_{t+1}$  will exactly satisfy the stated relationship provided the explanatory variables and sample period are the same for both variables, as we of course assume.

<sup>19</sup>Evans and Honkapohja (2006) derive the Euler equations for the general equilibrium framework of Woodford (1996).

Preston (2006) analyzes optimal monetary policy under commitment from the timeless perspective considered in Ch. 7 of Woodford (2003). Preston (2006) looks at determinate monetary policies capable of implementing the optimal equilibrium under IH learning dynamics as in Preston (2005). He examines variants of monetary policies that respond to (one-step ahead) future forecasts of inflation and output gap (capable of implementing the optimal equilibrium), similar to the Taylor type rules considered in Bullard and Mitra (2002).

The IH approach used in Preston requires agents to forecast all future paths of nominal interest rates (along with the forecasts of inflation and output gap) while the EE approach used in Bullard and Mitra (2002) does not require agents to do so (since no forecasts of interest rates appear in the Euler equations). Another way to interpret these differences is to assume that agents do not know the policy rule being used by the central bank in Preston (2006) while they have this knowledge in Bullard and Mitra (2002) (say due to the central bank being more transparent about its policy). Preston (2006) claims that the results on convergence of learning dynamics can be different between the IH and EE approaches under these different informational assumptions; see his Proposition 2. On the other hand, if agents have knowledge of the monetary policy rule being used by the central bank (as assumed in Bullard and Mitra (2002)), then Preston (2006) continues to find *exactly* the same conditions determining stability under learning dynamics for his IH model; see his Proposition 3.

These results are perhaps not that surprising since it is well known in the adaptive learning literature that the conditions for stability under learning dynamics depend crucially on the form of the perceived law of motion (PLM) used by the economic agents. Stability conditions for the same economic model can vary depending on the nature of the PLMs used by agents (see Evans and Honkapohja (2001) for a number of examples). Depending on whether the agents have knowledge of the monetary policy rule or not will lead to different PLMs and can affect stability conditions in the monetary model considered above.<sup>20</sup>

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<sup>20</sup>Similar remarks apply to the IH approach used in Preston (2008) where he analyzes the expectations-based reaction function proposed by Evans and Honkapohja (2006) which uses the

## References

- BRANCH, W. A., G. W. EVANS, AND B. MCGOUGH (2010): “Finite Horizon Learning,” mimeo.
- BULLARD, J., AND K. MITRA (2002): “Learning About Monetary Policy Rules,” *Journal of Monetary Economics*, 49, 1105–1129.
- EVANS, G. W., AND S. HONKAPOHJA (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press, Princeton, New Jersey.
- (2003): “Expectations and the Stability Problem for Optimal Monetary Policies,” *Review of Economic Studies*, 70, 807–824.
- (2006): “Monetary Policy, Expectations and Commitment,” *Scandinavian Journal of Economics*, 108, 15–38.
- EVANS, G. W., S. HONKAPOHJA, AND K. MITRA (2009): “Anticipated Fiscal Policy and Learning,” *Journal of Monetary Economics*, 56, 930–953.
- EVANS, G. W., AND B. MCGOUGH (2010): “Learning to Optimize,” mimeo.
- HONKAPOHJA, S., AND K. MITRA (2005): “Performance of Monetary Policy with Internal Central Bank Forecasting,” *Journal of Economic Dynamics and Control*, 29, 627–658.
- MARCET, A., AND T. J. SARGENT (1989): “Convergence of Least-Squares Learning Mechanisms in Self-Referential Linear Stochastic Models,” *Journal of Economic Theory*, 48, 337–368.
- PRESTON, B. (2005): “Learning about Monetary Policy Rules when Long-Horizon Expectations Matter,” *International Journal of Central Banking*, 1, 81–126.
- (2006): “Adaptive Learning, Forecast-based Instrument Rules and Monetary Policy,” *Journal of Monetary Economics*, 53, 507–535.

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EE approach ; see Proposition 5 of Preston (2008).

- (2008): “Adaptive Learning and the Use of Forecasts in Monetary Policy,” *Journal of Economic Dynamics and Control*, 32, 3661–3681.
- SARGENT, T. J. (1993): *Bounded Rationality in Macroeconomics*. Oxford University Press, Oxford.
- WOODFORD, M. (1996): “Control of the Public Debt: A Requirement for Price Stability?,” Working paper, NBER WP5684.
- (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press, Princeton, NJ.

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